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A group theoretic approach to a class of second-order ordinary differential equations not possessing Lie point symmetries

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Abstract. We consider the solution of a class of second-order ordinary differential equations not possessing Lie point symmetries by group theoretic means. The method involves increasing the order of these equations using homogeneity symmetry. The solution of this new third-order equation is then sought in the instances that it and the new reduced second-order equation possess additional symmetries. As a result the number of second-order equations solvable by the Lie theory of extended groups is increased.

1. Introduction

In Lie's development of the theory of the application of transformation groups [1–3] to differential equations [4] he strove to provide a unified treatment of the methods of their solution which up until then had been very much *ad hoc*. While Lie embraced the concept of the unification of methods, his classical theory does not, unfortunately, apply to all equations. In some instances the group approach fails where elementary methods succeed. In this paper, as in our recent work on nonlocal symmetries [5], we wish to extend and supplement Lie's results by increasing the number and variety of second-order ordinary differential equations for which a solution is possible, we will do this by group theoretic means.

The inspiration for what we report here is the solution of the equation

$$z'' = \frac{z'^2}{z} + a(x)zz' + a'(x)z^2 \tag{1.1}$$

which was considered by González-Gascón and Gonzáles-López [6], Vawda [7] and Abraham-Shrauner *et al* [8]. We use the latter's method since it provides a simple instance of the theory to be elaborated below.

The order of (1.1) is increased by the generalized Riccati transformation

$$z = -\frac{w'}{aw} \tag{1.2}$$

and it becomes

$$w'w''' - w''^2 - \left(\frac{a'}{a}\right)' w'^2 = 0. \tag{1.3}$$

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Equation (1.3) has only the two obvious Lie point symmetries

$$G_1 = \frac{\partial}{\partial w} \qquad G_2 = w\frac{\partial}{\partial w} \tag{1.4}$$

(confirmed using LIE [9]). Transformation (1.2) corresponds to the use of G_2 to reduce the third-order equation (1.3), to the second-order equation (1.1). The symmetry, G_1, is lost (as a point symmetry) in this reduction since G_2 is the nonnormal subgroup [10, p 148]. If we use G_1 to reduce the order of (1.3), the transformation

$$W = \log\frac{w'}{a} \qquad X = x \tag{1.5}$$

leads to

$$\frac{\mathrm{d}^2W}{\mathrm{d}X^2} = 0 \tag{1.6}$$

which not only is easy to solve but also possesses eight Lie point symmetries [4, p 404]. The solution to the original equation follows a reversal of the several transformations.

This example contains the essence of the method which is to be developed in generality in this paper. The central principle is to increase the order of a differential equation by a transformation which produces a point symmetry in the higher-order equation. Thus, transformation (1.2) automatically makes G_2 a symmetry of (1.3). As G_2 is not a 'good' symmetry to use to reduce the order, there is the possibility that the third-order equation has the symmetry G_1 as well.

The use of the generalized Riccati transformation (1.2) is motivated by its success in transforming the first-order nonlinear generalized Riccati equation [11, p 23]

$$y' + a(x) + b(x)y + c(x)y^2 = 0 \tag{1.7}$$

to the second-order linear equation

$$z'' + \left(b - \frac{c'}{c}\right)z' + acz = 0 \tag{1.8}$$

via

$$y(x) = \frac{z'(x)}{c(x)z(x)} \tag{1.9}$$

without a knowledge of the solution to (1.8). Transformation (1.9) is associated with the homogeneity symmetry, G_2, for which appropriate variables are found from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}w}{w} = \frac{\mathrm{d}w'}{w'}. \tag{1.10}$$

The invariants of (1.10) are

$$u = x \qquad v = \frac{w'}{w} \tag{1.11}$$

and so suitable new variables are

$$X = F\left(x, \frac{w'}{w}\right) \qquad W = G\left(x, \frac{w'}{w}\right) \tag{1.12}$$

where F and G are independent functions. In the example used such generality was not necessary.

The constraint placed on the original second-order equation is that the third-order equation has two symmetries so that there is an alternate route to reduce the third-order equation to second order. The aim of the present study is to determine which types of this class of new second-order equations will lead to solvable examples of the original second-order equation which are not trivially solvable. The best approach is to start with a third-order equation which is required to have two Lie point symmetries and that one of them persists in the second-order equation obtained by reduction using the normal subgroup. (We believe that the second-order equation we are interested in was obtained by the reduction of this third-order equation via the nonnormal subgroup and so will not have any Lie point symmetries.) As the existence of one point symmetry is not sufficient for integrability of the new second-order equation we determine those which have two point symmetries according to the classification of two-dimensional algebras by Lie [4, p 412]. Note that the existence of two point symmetries at the second-order level does not imply the existence of three point symmetries at the third-order level. The second symmetry *could* be a type II hidden symmetry [12]. Although the two-dimensional algebra is sufficient for integrability we provide a listing of the second-order equations with three and eight point symmetries which fit into our scheme. For all of these we give the representative second-order equation with no point symmetries from which the process commences. These will comprise the new classes of second-order equations solvable by the Lie method even though they do not possess Lie point symmetries.

2. General form of the third-order equation

We require that the third-order equation is invariant under the action of a two-dimensional algebra and that, due to the origin of the equation, one of the symmetries is the homogeneity symmetry

$$G_2 = y\frac{\partial}{\partial y}. \tag{2.1}$$

There are four two-dimensional transitive Lie algebras of vector fields in $\Re^2$ [4, p 412]. We are not interested in the two Abelian algebras

$$\begin{array}{llll} \text{type I} & G_1 = \dfrac{\partial}{\partial X} & G_2 = \dfrac{\partial}{\partial Y} & [G_1, G_2] = 0 \\ \text{type II} & G_1 = \dfrac{\partial}{\partial Y}] & G_2 = X\dfrac{\partial}{\partial Y} & [G_1, G_2] = 0 \end{array} \tag{2.2}$$

since both elements are normal subgroups and reduction by one does not mean the loss of the other as a point symmetry of the reduced equation [10, p 148]. In our approach the original second-order equation can be imagined as coming from the third-order equation as a result of reduction using G_2 (in (2.1)). The other symmetry is lost because G_2 is not the normal subgroup. Our method then utilizes the normal subgroup to reduce the third-order equation to another second-order equation. This equation will inherit G_2. To solve the new second-order equation we seek those equations which admit a second Lie point symmetry. This procedure imposes the minimum constraint on the third-order equation. We do not require that it has a third point symmetry, only that it has a nonlocal symmetry which becomes point symmetry in the variables of the reduced equation.

The two solvable algebras (types III and IV of Lie's classification [4, p 425]) have the canonical forms

$$\begin{array}{llll} \text{type III} & G_1 = \dfrac{\partial}{\partial Y} & G_2 = X\dfrac{\partial}{\partial X} + Y\dfrac{\partial}{\partial Y} & [G_1, G_2] = G_1 \\ \text{type IV} & G_1 = \dfrac{\partial}{\partial Y} & G_2 = Y\dfrac{\partial}{\partial Y} & [G_1, G_2] = G_1. \end{array} \tag{2.3}$$

We shall determine the class of third-order equations invariant under each canonical realization and then transform the nonnormal subgroup to the desired form of G_2, namely (2.1). In an attempt to avoid confusion we use the subscripts '2' and '3' to denote the two-dimensional Lie algebras associated with the second- and third-order equations respectively.

We recall that the third-order ordinary differential equation (in solved form)

$$y''' = f(x, y, y', y'') \tag{2.4}$$

possesses a Lie point symmetry,

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.5}$$

if

$$G^{[3]}(y''' - f)_{|_{(y'''-f)=0}} = 0 \tag{2.6}$$

where

$$\begin{aligned} G^{[3]} = G &+ (\eta' - y'\xi')\frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''} \\ &+ (\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi''')\frac{\partial}{\partial y'''}. \end{aligned} \tag{2.7}$$

It is also possible to use (2.5) to determine those functions, f, for which (2.4) possesses the symmetry (2.5) by calculating the invariants of the symmetry. We consider each canonical realization in turn.

2.1. Type III_3

We begin with type III_3 two-dimensional algebra in (2.3). It is evident that the form of (2.4) invariant under G_1 is

$$Y''' = f(X, Y', Y''). \tag{2.8}$$

Under G_2 the function, f, satisfies the first-order linear partial differential equation

$$X\frac{\partial f}{\partial X} - Y''\frac{\partial f}{\partial Y''} = -2f \tag{2.9}$$

with associated Lagrange's system

$$\frac{\mathrm{d}X}{X} = \frac{\mathrm{d}Y'}{0} = \frac{\mathrm{d}Y''}{-Y''} = \frac{\mathrm{d}f}{-2f}. \tag{2.10}$$

The characteristics are

$$\begin{aligned} u &= XY'' \\ v &= Y' \\ w &= \frac{f}{Y''^2} \end{aligned} \tag{2.11}$$

and (2.8) becomes

$$Y''' = Y''^2 f(Y', XY''). \tag{2.12}$$

Under the transformation

$$x = F(X, Y) \qquad y = G(X, Y) \tag{2.13}$$

G_2 takes the required form (2.1) if

$$F = A\left(\frac{Y}{X}\right) \qquad G = YB\left(\frac{Y}{X}\right) \tag{2.14}$$

where A and B are arbitrary functions of their arguments. The structure of transformation (2.14) suggests a happy simplification. If we take A to be the identity and B to be one over the identity, transformation (2.14) is

$$x = \frac{Y}{X} \qquad y = X \tag{2.15}$$

which represents all classes of equations up to the transformation

$$x \longrightarrow a(x) \qquad y \longrightarrow yb(x). \tag{2.16}$$

Under transformation (2.15), (2.12) becomes

$$y''' = 3y''\left(\frac{yy'' - y'^2}{yy'}\right) - y\left(\frac{yy'' - 2y'^2}{yy'}\right)^2 f\left[\frac{xy' + y}{y'}, -\frac{y^2}{y'^2}\left(\frac{yy'' - 2y'^2}{yy'}\right)\right] \tag{2.17}$$

or, equivalently,

$$y''' = 3y''\left(\frac{yy'' - y'^2}{yy'}\right) - \frac{y'^4}{y^3} f\left[\frac{xy' + y}{y'}, -\frac{y^2}{y'^2}\left(\frac{yy'' - 2y'^2}{yy'}\right)\right]. \tag{2.18}$$

The symmetry G_1 is now

$$G_1 = \frac{1}{y}\frac{\partial}{\partial x}. \tag{2.19}$$

2.2. Type IV$_3$

In a similar manner we find the third-order equation invariant under the canonical representation of type IV$_3$ algebra in (2.3), namely

$$Y''' = Y'' f\left(X, \frac{Y''}{Y'}\right). \tag{2.20}$$

As G_2 is in the desired form, the only admissible transformations are those which transform it to itself. These are of the class (2.16). Hence, up to this equivalence class the normal form of the equation is

$$y''' = y'' f\left(x, \frac{y''}{y'}\right) \tag{2.21}$$

and

$$G_1 = \frac{\partial}{\partial y}. \tag{2.22}$$

While both (2.12) and (2.21) are given in [13] we present the above detail to aid the reader in applying the method to equations other than second order.

3. The reduced second order equation: type III_3 Lie algebra

We first consider the reduction of the third-order equation invariant under type III_3 two-dimensional Lie algebra, namely (2.18). The transformation which the symmetry (2.19) naturally suggests is

$$u = y \qquad v = x + \frac{y}{y'} \tag{3.1}$$

which reduces (2.18) to the second-order equation

$$u^2 v'' = f(v, uv'). \tag{3.2}$$

The symmetry is

$$G = u\frac{\partial}{\partial u}. \tag{3.3}$$

Neither equation nor symmetry are in the most suitable form. We make the further transformation

$$U = v \qquad V = \log u \qquad g(U, V') = -V'^3 f\left(U, \frac{1}{V'}\right) - V'^2 \tag{3.4}$$

and obtain

$$\frac{\mathrm{d}^2 V}{\mathrm{d}U^2} = g(U, V') \tag{3.5}$$

which has the symmetry

$$G = \frac{\partial}{\partial V} \tag{3.6}$$

so that we can make a direct comparison with the four types of equations with two symmetries (cf table 1 with $U \leftrightarrow u$ and $V \leftrightarrow v$). equations invariant under the type I_2 algebra must have g free of U which means that (2.18) is autonomous and so has the extra symmetry $\partial/\partial x$ which is not lost under the reduction of order using $y\partial/\partial y$. Hence, the original second-order equation has one symmetry and is not within the class considered here.

For (3.5) to be invariant under type II_2 it must be free of V' which means that

$$f\left(U, \frac{1}{V'}\right) = -\frac{F(U) + V'^2}{V'^3}. \tag{3.7}$$

Table 1. Canonical forms of $v'' = g(u, v')$ admitting two Lie point symmetries.

Type	$[G_1, G_2]$	Canonical forms of G_1 and G_2	Form of equation
I_2	0	$G_1 = \frac{\partial}{\partial u}$ $G_2 = \frac{\partial}{\partial v}$	$v'' = F(v')$
II_2	0	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial v}$	$v'' = F(u)$
III_2	G_1	$G_1 = \frac{\partial}{\partial v}$ $G_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$	$uv'' = F(v')$
IV_2	G_1	$G_1 = \frac{\partial}{\partial v}$ $G_2 = v\frac{\partial}{\partial v}$	$v'' = v'F(u)$

The third-order equation takes the form

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'^3}{y^2}\left(2 - \frac{yy''}{y'^2}\right) + y'\left(2 - \frac{yy''}{y'^2}\right)^3 F\left(x + \frac{y}{y'}\right). \tag{3.8}$$

Equation (3.8) has just two point symmetries and so the original second-order equation obtained by means of the Riccati transformation

$$z = x \qquad w = \frac{y'}{y} \tag{3.9}$$

namely

$$w'' = \frac{3w'^2}{w} - ww' + w\left(1 - \frac{w'}{w^2}\right)^3 F\left(z + \frac{1}{w}\right) \tag{3.10}$$

has no symmetry for nontrivial F.

In the case of type III_2 algebra the function in (3.5) has a very specific U dependence [14], namely

$$g(U, V') = \frac{1}{U} F(V') \tag{3.11}$$

and so

$$f\left(U, \frac{1}{V'}\right) = -\frac{1}{V'^3}\left(\frac{1}{U} F(V') + V'^2\right) \tag{3.12}$$

$$f(v, uv') = -(uv')^3\left(\frac{1}{v} F\left(\frac{1}{uv'}\right) + \frac{1}{(uv')^2}\right). \tag{3.13}$$

The third-order equation is

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'}{y^2}(2y'^2 - yy'') + \frac{(2y'^2 - yy'')^3}{y'^4(xy' + y)} F\left(\frac{y'^3}{y(2y'^2 - yy'')}\right) \tag{3.14}$$

and the original second-order equation (via (3.9)) is

$$w'' = \frac{3w'^2}{w} - ww' + \frac{w^2 - w'}{w^4(1 + zw)} F\left(\frac{w^3}{w^2 - w'}\right). \tag{3.15}$$

The third-order equation has just the two symmetries for general $F(V')$. There is a technical difficulty with this case in that the integration of

$$UV'' = F(V') \tag{3.16}$$

will give V' as an implicit function of U. In principle this is not a problem, but there are definitely going to be practical difficulties for an unspecified $F(V')$. The solution to this problem is addressed in part in section 5.1 where we consider the three-dimensional and eight-dimensional symmetry cases for type III_2 in both this and type IV_3 realization.

Type IV_2 is linear and the solution readily expressed as a quadrature. From (3.4)

$$f\left(U, \frac{1}{V'}\right) = -\frac{F(U)}{V'^2} - \frac{1}{V'} \tag{3.17}$$

and the third-order equation is

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'^3}{y^2}\left(2 - \frac{yy''}{y'^2}\right) + \frac{y'^2}{y}\left(2 - \frac{yy''}{y'^2}\right)^2 F\left(x + \frac{y}{y'}\right). \tag{3.18}$$

The general dependence of F on U which contains x means that for arbitrary F there is not an additional symmetry and the original second-order equation

$$w'' = \frac{3w'^2}{w} - ww' + \left(\frac{w'}{w} - w\right)^2 F\left(z + \frac{1}{w}\right) \tag{3.19}$$

obtained by the Riccati reduction (3.9) will not have a point symmetry.

4. The reduced second-order equation: type IV_3 Lie algebra

We now turn our attention to the third-order equation invariant under type IV_3 two-dimensional Lie algebra, namely (2.21). The standard transformation for the reduction of the order of (2.21) with the symmetry (2.22) is to set

$$u = x \qquad v = y'. \tag{4.1}$$

However, the reduced equation is made a little simpler in appearance if the transformation

$$u = x \qquad v = \log y' \qquad g(u, v') = v' f(u, v') - v'^2 \tag{4.2}$$

is used, as then (2.21) becomes

$$v'' = g(u, v'). \tag{4.3}$$

In the new coordinates, G_2 becomes the symmetry

$$X_1 = \frac{\partial}{\partial v}. \tag{4.4}$$

It then becomes a very simple matter to read off the canonical forms of (4.3) under the four algebras of two dimensions since the symmetry (4.4) appears in each one of them. We summarize these algebras in table 1.

For type I_2 the canonical form has g independent of u and (2.21) has the additional symmetry

$$G_3 = \frac{\partial}{\partial x} \tag{4.5}$$

which has zero Lie bracket with G_2. Hence the original second-order equation has at least one symmetry and is not of the class sought. For type II_2

$$f(u, v') = \frac{F(u)}{v'} + v' \tag{4.6}$$

and (2.21) has the form

$$y''' = y' \left\{ \left(\frac{y''}{y'}\right)^2 + F(x) \right\}. \tag{4.7}$$

For general $F(x)$ (4.7) has no additional symmetry. Type II_2 canonical form has eight symmetries and is integrable by quadrature. Hence, the solution of the original second-order equation derived from (4.7) by reduction using (3.9), namely

$$w'' = \frac{w'^2}{w} - ww' + wF(z) \tag{4.8}$$

is always integrable. Any equation related to (4.8) by an invertible point transformation is also integrable. In fact, if we set

$$w = a(Z)W \qquad z = -Z \tag{4.9}$$

we obtain (1.1).

For type III_2 the actual solution to the equation requires an inversion after an integration. It is best to look at the particular instances in which this is possible (see section 5.2). For now we give the third-order equation, namely

$$y''' = y'\left\{\frac{1}{x}F\left(\frac{y''}{y'}\right) + \left(\frac{y''}{y'}\right)^2\right\} \tag{4.10}$$

and the original second-order equation via (3.9), namely

$$w'' = \frac{w'^2}{w} - ww' + \frac{w}{z}F\left(\frac{w'}{w} + w\right). \tag{4.11}$$

In the case of a type IV_2 equation the solution follows from a straightforward quadrature. The third-order equation becomes

$$y''' = y'\left[\left(\frac{y''}{y'}\right)^2 + \frac{y''}{y'}F(x)\right] \tag{4.12}$$

and for general $F(x)$ this has no other point symmetries than G_1 and G_2. Reduction by the Riccati transformation gives an integrable second-order equation, namely

$$w'' = \frac{w'^2}{w} - ww' + (w' + w^2)F(z) \tag{4.13}$$

which has no point symmetries and is representive of a whole class of equations equivalent to it under a point transformation.

5. The reduced second-order equation: type III_2 Lie algebra

For both type III_3 and type IV_3 symmetries of the third-order equation we have seen that suitable equations, i.e. ones without point symmetry, can lead via the increase in order with the use of the Riccati transformation and the reduction via the normal subgroup to equations which have either eight symmetries (types II_2 and IV_2) and are readily reduced to quadratures or equations which have at least two point symmetries (type III_2). As we expect some difficulties in the integration of this equation, we will look at the special cases of type III_2 equations which have either three or eight symmetries.

The equations which have three symmetries are going to be either of type I_2 or type III_2 with some constraint due to the extra symmetry. As type I_2 equations do not lead to the case of the second-order equation not possessing symmetry, we need to consider only those equations which belong to type III_2. There are only four equivalence classes of equations of type III_2 [15] with three symmetries and we list them together with their symmetries in table 2.

5.1. Type III_3

We consider type III_3 reduction first. (Note that the variables in table 2 are read the same way as those for table 1 in section 3.) We recall that the third-order equation is (3.14), namely

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'}{y^2}(2y'^2 - yy'') + \frac{(2y'^2 - yy'')^3}{y'^4(xy' + y)}F\left(\frac{y'^3}{y(2y'^2 - yy'')}\right) \tag{5.1}$$

where $F(\cdot)$ is one of the functions listed in table 2. The symmetry G_1 in table 2 corresponds to the symmetry G_2 (in (2.1)) which is to be used to reduce the third-order equation to the

Table 2. Equivalence classes of equations of type III_2 with an additional symmetry. (In each case $K \neq 0$.)

Equation	Symmetries	Algebra
$uv'' = v'^3 + v' + K(1+v'^2)^{3/2}$	$G_1 = \dfrac{\partial}{\partial v}$	$A_{3,8}(s\ell(2,R))$
	$G_2 = u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v}$	$[G_1, G_2] = G_1$
	$G_3 = 2uv\dfrac{\partial}{\partial u} + (v^2 - u^2)\dfrac{\partial}{\partial v}$	$[G_1, G_3] = 2G_2$
		$[G_2, G_3] = G_3$
$uv'' = Kv'^3 - \frac{1}{2}v'$	$G_1 = \dfrac{\partial}{\partial v}$	$A_{3,8}(s\ell(2,R))$
	$G_2 = u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v}$	$[G_1, G_2] = G_1$
	$G_3 = 2uv\dfrac{\partial}{\partial u} + v^2\dfrac{\partial}{\partial v}$	$[G_1, G_3] = G_2$
		$[G_2, G_3] = G_3$
$uv'' = (a-1)v' + Kv'^{\frac{2a-1}{a-1}}$	$G_1 = \dfrac{\partial}{\partial v}$	$A_{3,5}^a$
$a \neq 0, \frac{1}{2}, 1, 2$	$G_2 = u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v}$	$[G_1, G_2] = G_1$
	$G_3 = u^{1-a}\dfrac{\partial}{\partial u}$	$[G_1, G_3] = 0$
		$[G_2, G_3] = (1-a)G_3$
$uv'' = 1 + K\exp(v')$	$G_1 = \dfrac{\partial}{\partial v}$	$A_{3,2}$
	$G_2 = u\dfrac{\partial}{\partial u} + v\dfrac{\partial}{\partial v}$	$[G_1, G_2] = G_1$
	$G_3 = \dfrac{\partial}{\partial u} + \log u\dfrac{\partial}{\partial v}$	$[G_1, G_3] = 0$
		$[G_2, G_3] = G_1 - G_3$

second-order equation with no point symmetries. The Lie brackets show that G_2 in table 2 is a possible symmetry of the original equation in all cases since G_1 is the normal subgroup for the pair. However, unravelling the various transformations at the third-order level we find that

$$G_2 = \left\{x + \frac{1}{y}\int y\mathrm{d}x\right\}\frac{\partial}{\partial x} + y\log y\frac{\partial}{\partial y} \tag{5.2}$$

which is nonlocal and reduction via $y\partial/\partial y$ leaves it that way. In the case of the two representations of $s\ell(2,R)$ in table 2 G_3 is not possible as a point symmetry of the original equation since $[G_1, G_3] = G_2$. However, in the cases of $A_{3,2}$ and $A_{3,5}^a$ it must be considered. In both cases the symmetry is nonlocal at the third order since for $A_{3,5}^a$

$$\xi = \frac{1}{y}\int y'\left(x + \frac{y}{y'}\right)^{1-a}\mathrm{d}x \qquad \eta = 0 \tag{5.3}$$

and for $A_{3,2}$

$$\xi = 1 - \frac{1}{y}\int\left(\eta - \frac{y}{y'}\eta'\right)\mathrm{d}x \qquad \eta = y\log\left(x + \frac{y}{y'}\right) \tag{5.4}$$

and the nonlocal property persists with the $y\partial/\partial y$ reduction. Thus, all four of the equivalence classes of type III_2 with additional symmetry do not produce a point symmetry in the original second-order equation. It is a simple matter to determine the original second-order equation in all four cases. We use (3.15) with F replaced by the expressions on the right-hand side of the equations in table 2 with v' replaced by the argument of F. The third-order equation is obtained in a similar manner using (5.1).

5.2. Type IV$_3$

The symmetry G_2 of table 2 becomes nonlocal at the third order and is

$$G_2 = x\frac{\partial}{\partial x} + \left(y + \int y'\log y'\,\mathrm{d}x\right)\frac{\partial}{\partial y}. \tag{5.5}$$

The nonlocal nature persists after reduction using $y\partial/\partial y$. For $A^a_{3,5}$

$$G_3 = x^{1-a}\frac{\partial}{\partial x} + \left\{(1-a)yx^{-a} + a(1-a)\int yx^{-a-1}\,\mathrm{d}x\right\}\frac{\partial}{\partial y} \tag{5.6}$$

at the third-order level and this remains nonlocal under reduction by $y\partial/\partial y$. For $A_{3,2}$

$$G_3 = \frac{\partial}{\partial x} + \left(y\log x - \int\frac{y}{x}\,\mathrm{d}x\right)\frac{\partial}{\partial y} \tag{5.7}$$

which also remains nonlocal when reduction via $y\partial/\partial y$ is performed. (Recall that $a \neq 0, 1$ and so the integral in (5.6) is always present.)

Concern was expressed in section 3 as to the feasibility of type III$_2$ equations because of problems of inversion. All integrals for these four classes can be inverted. The original second-order equation is (4.11) with F replaced by the expressions on the right-hand side of the equations in table 2 with v' replaced by the argument of F. The third-order equation is obtained in a similar manner via (4.10).

5.3. Linear equations

The final case to consider is when type III$_2$ admits eight symmetries. The representative equation is [15]

$$uv'' = v'^3 + v' \tag{5.8}$$

with the symmetries

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial v}\\
G_2 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\\
G_3 &= \left(u + \frac{v^2}{u}\right)\frac{\partial}{\partial u}\\
G_4 &= \frac{1}{u}\frac{\partial}{\partial u}\\
G_5 &= \frac{v^3}{u}\frac{\partial}{\partial u} - \frac{1}{2}(u^2+3v^2)\frac{\partial}{\partial v}\\
G_6 &= \left(\frac{v^4}{4u} - \frac{u^3}{4}\right)\frac{\partial}{\partial u} - \frac{1}{2}\left(vu^2 + v^3\right)\frac{\partial}{\partial v}\\
G_7 &= \frac{v}{u}\frac{\partial}{\partial u}\\
G_8 &= \left(uv - \frac{v^3}{u}\right)\frac{\partial}{\partial u} + 2v^2\frac{\partial}{\partial v}.
\end{aligned} \tag{5.9}$$

The Lie brackets with G_1 are

$$[G_1, G_2] = G_1 \qquad [G_1, G_3] = 2G_7 \qquad [G_1, G_4] = 0$$
$$[G_1, G_5] = -3(G_2 - G_3) \qquad [G_1, G_6] = G_5 \qquad [G_1, G_7] = G_4 \tag{5.10}$$
$$[G_1, G_8] = 4G_2 - 3G_3$$

and so possible candidates are just G_2 and G_4. However, we have already seen that G_2 does not lead to a point symmetry and so we need only consider G_4. For type III_3 equations we find that at the third-order level

$$G_4 = \frac{1}{y}\int \frac{y'^2\,\mathrm{d}x}{xy' + y}\frac{\partial}{\partial x} \tag{5.11}$$

and consequently is nonlocal under the reduction via $y\frac{\partial}{\partial y}$. The third-order equation is

$$y''' = 3y''\left(\frac{y''}{y'} - \frac{y'}{y}\right) + \frac{y'}{y^2}(2y'^2 - yy'') + \frac{(2y'^2 - yy'')^3}{y'^4(xy' + y)}\left[\left(\frac{y'^3}{y(2y'^2 - yy'')}\right)^3 + \frac{y'^3}{y(2y'^2 - yy'')}\right] \tag{5.12}$$

and the original second-order equation is

$$w'' = \frac{3w'^2}{w} - ww' + \frac{w^2 - w'}{w^4(1 + zw)}\left[\left(\frac{w^3}{w^2 - w'}\right)^3 + \frac{w^3}{w^2 - w'}\right]. \tag{5.13}$$

For type IV_3 equations we find that G_4 becomes

$$G_4 = \frac{1}{x}\frac{\partial}{\partial x} + \left\{-\frac{y}{x^2} - 2\int \frac{y}{x^3}\,\mathrm{d}x\right\}\frac{\partial}{\partial y} \tag{5.14}$$

which remains nonlocal when the reduction via $y\frac{\partial}{\partial y}$ is performed. The third-order equation is

$$y''' = y'\left\{\frac{1}{x}\left[\left(\frac{y''}{y'}\right)^3 + \frac{y''}{y'}\right] + \left(\frac{y''}{y'}\right)^2\right\} \tag{5.15}$$

and the orginal second-order equation is

$$w'' = \frac{w'^2}{w} - ww' + \frac{w}{z}\left\{\left(\frac{w' + w^2}{w}\right)^3 + \frac{w' + w^2}{w}\right\}. \tag{5.16}$$

Hence, in all cases type III_2 equations lead to no point symmetry in the original second-order equation.

6. Discussion

The Lie method of extended groups is attractive in that it provides an algorithmic method to solve differential equations. However, in the instances when the equations being studied do not possesses Lie point (contact) symmetries the method is inapplicable. We have shown that, by reversing the standard procedure, some progress can still be made. Due to their proliferation in applications we have concentrated on second-order equations and presented those which do not have Lie point symmetries yet are solvable using the Lie method. The method applies *mutatis mutandis* to higher-order equations.

In this work we have considered all two-dimensional algebras of symmetries. We have thereby ignored the fact that the second-order equations invariant under types II_2 and IV_2 are linear and hence have eight Lie point symmetries. Thus, there exists a point transformation to take the second-order equation invariant under type II_2 to that invariant under type IV_2. This suggests a relationship between the original second-order equations not possessing Lie point symmetries. In the case of (4.8) the transformation

$$w = W \int F(z)\,\mathrm{d}z \qquad z = \int \frac{1}{\int F(z)}\,\mathrm{d}z \tag{6.1}$$

yields (4.13) (with w in (4.13) replaced by W). A similar result will hold for (3.10) and (3.19).

We note that the solutions to equations (4.8) and (4.13) are easily obtained. Both integrate trivially to a first integral that can be rewritten as a Riccati equation—this integration is related to the presence of exponential nonlocal symmetries of the form [8]

$$G = \exp\left[\int P(x, y)\,\mathrm{d}x\right]\left(\xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}\right). \tag{6.2}$$

(Thus, the route to the linear second-order equation is obtainable by this elementary method and that detailed earlier.) The reason we still consider the solution of these equations via our method is to show that those equations which are solvable by elementary methods are so because of the group theoretic basis. However, the practical usefulness of the approach is evidenced by the solutions of (3.10), (3.15), (3.19) and (4.11) which are not obviously integrable.

The work presented here opens up a number of new avenues. The first is the extension to second-order equations possessing one Lie point symmetry. Here, one would require that the third-order equations possess three Lie point symmetries and that reduction using the third symmetry results in a new second-order equation with more than one symmetry. A further aspect that would need to be investigated in this case is that of the original second-order equation possessing, in addition to the one Lie point symmetry, a ‘useful’ nonlocal symmetry, i.e. a nonlocal symmetry that reduces to a new point symmetry under the reduction of the second-order equation via the point symmetry [5]. In fact that option can also be applied to the new second-order equations in this paper—it is not necessary for these equations to possess two Lie point symmetries to be reducible to quadratures.

Another avenue of research is the possession of contact symmetries by the third-order equation. As it has been shown [16] that contact symmetries can reduce to point symmetries, this could result in new classes of third-order equations that reduce to second-order equations which are solvable without the imposition of further restrictions, i.e. they may naturally possess more than one Lie point symmetry.

Contact symmetries can also be considered for original second-order equations. While there are technical difficulties associated with finding these contact symmetries (essentially the equation has to be solved to obtain them), it is possible to assume that the second-order equation has no Lie point symmetries but one contact symmetry of some specifc form. Then an increase of order could lead to a third-order equation with more than one Lie point symmetry etc.

From above it can be seen that the ideas presented here can be used successfully to extend the number of solvable equations. They can also be used to explain the integration of equations not possessing Lie point symmetries via group theory. It is worth noting that we do not expect ‘fundamental equations’ such as the six Painlevé equations [17–19] to be solvable using our method—these equations are ‘irreducible’ [11, p 345]. It is hoped that these endeavours further enlarge the classes of equations solvable via the Lie method and bring us closer to realizing Lie’s ideal of the solution of all differential equations in a unified manner.

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References

[1] Lie S 1970 *Theorie der Transformationsgruppen* vol I, 2nd edn (New York: Chelsea)
[2] Lie S 1970 *Theorie der Transformationsgruppen* vol II, 2nd edn (New York: Chelsea)
[3] Lie S 1970 *Theorie der Transformationsgruppen* vol III, 2nd edn (New York: Chelsea)
[4] Lie S 1967 *Differentialgleichungen* (New York: Chelsea)
[5] Govinder K S and Leach P G L 1995 On the determination of nonlocal symmetries *J. Phys. A: Math. Gen.* **28** 5349–59
[6] González-Gascón F and González-López A 1988 Newtonian systems of differential equations integrable via quadratures, with trivial group of point symmetries *Phys. Lett.* A **129** 153–6
[7] Vawda F E 1994 An application of Lie analysis to classical mechanics *MSc Dissertation* University of the Witwatersrand, Johannesburg
[8] Abraham-Shrauner B, Govinder K S and Leach P G L 1995 Integration of second order equations not possessing point symmetries *Phys. Lett.* A **203** 169–74
[9] Head A K 1993 LIE, a PC program for Lie analysis of differential equations *Comput. Phys. Commun.* **77** 241–8
[10] Olver P J 1993 *Applications of Lie Groups to Differential Equations (Graduate Texts in Mathematics 107)* 2nd edn (New York: Springer)
[11] Ince E L 1927 *Ordinary Differential Equations* (London: Longmans)
[12] Abraham-Shrauner B and Guo A 1992 Hidden symmetries associated with the projective group of nonlinear first order ordinary differential equations *J. Phys. A: Math. Gen.* **25** 5597–608
[13] Gat O 1992 Symmetry algebras of third-order ordinary differential equations *J. Math. Phys.* **33** 2966–71
[14] Mahomed F M 1986 Classification of nonlinear ordinary differential equations according to their symmetries *MSc Dissertation* University of the Witwatersrand, Johannesburg
[15] Mahomed F M 1989 Symmetry Lie algebras of nth order ordinary differential equations *PhD Thesis* University of the Witwatersrand, Johannesburg
[16] Abraham-Shrauner B, Leach P G L, Govinder K S and Ratcliff G 1995 Hidden and contact symmetries of ordinary differential equations *J. Phys. A: Math. Gen.* **28** 6707–16
[17] Painlevé P 1900 Mémoire sur les équations différentielles dont l'intégrale générale est uniforme *Bull. Math. Soc. France* **28** 201–61
[18] Painlevé P 1902 Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme *Acta Math.* **25** 1–85
[19] Gambier B 1909 Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est a points critiques fixes *Acta Math.* **33** 1–55